J. Fluid Mech. (1994), *vol.* 272, *pp.* 157–182

On Chaplygin's investigations of two-dimensional vortex structures in an inviscid fluid

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(Received 12 July 1993 and revised form 21 January 1994)

This paper describes exact solutions of two-dimensional vortex structures that were published by Chaplygin (1899, 1903) at the turn of the last century, which seem to have escaped the attention of later investigators in this field. Chaplygin's solutions include that of an elliptical patch of uniform vorticity in an exterior field of pure shear and that of a (symmetric or non-symmetric) dipolar vortex with a continuous distribution of vorticity translating steadily along a straight path. In addition, a solution is presented for a non-symmetric vortex dipole moving along a circular trajectory. A concise account of Chaplygin's solutions is given, complemented by a more detailed analysis of some of their relevant properties.

1. Introduction

During the last decade there has been an increasing interest in investigating the dynamics of inviscid two-dimensional coherent vortex structures under different conditions. Apart from offering important fundamental problems, coherent vortices are also believed to be relevant to the large-scale geophysical flows (see e.g. Flierl 1987). The dynamics of coherent vortex structures has been studied in the laboratory in a variety of configurations, ranging from a rotating fluid (e.g. Griffiths & Linden 1981; Flierl, Stern & Whitehead 1983; Kloosterziel & van Heijst 1991, see also the review article by Hopfinger & van Heijst 1993) or a stratified fluid (van Heijst & Flór 1989 *a*, *b*; Voropayev, Afanasyev & Filippov 1991) to a shallow layer of mercury subjected to a magnetic field (Nguyen Duc & Sommeria 1988) or even a soap film (Couder & Basdevant 1986). These experimental studies are all relatively recent.

On the other hand, the theoretical analysis of solutions of the Euler equations that represent the dynamics of isolated regions of distributed vorticity in a two-dimensional flow domain have attracted the attention of quite a few investigators since the mid 19th century. A detailed survey of the results obtained up to the beginning of this century can be found in the classical treatises by Basset (1888) and Lamb (1895, 1906), and also in the review articles by Love (1887, 1901) and Auerbach (1908). However, the number of exact solutions to the nonlinear Euler equations is limited. For patches of uniform vorticity in an ambient potential flow region, the circular Rankine vortex (Rankine 1858, Art. 633) and the elliptical Kirchhoff vortex (Kirchhoff 1876, Lecture XX) are such exact solutions. The Rankine vortex model was used by Maxwell (1861 *a*, *b*) in his early attempts to describe the nature of electrical and magnetic phenomena (he referred to these structures as 'molecular vortices'). In contrast to the Rankine vortex which represents a stationary vortical flow, the flow associated with the Kirchhoff vortex is non-steady, since the elliptical patch performs a steady rotation about its centre. During such a rotation, the Kirchhoff vortex preserves its elliptical shape.

Relatively recently, exact solutions were formulated for an elliptical patch of uniform vorticity embedded in an external strain and shear flow (Moore & Saffman 1971; Kida 1981). Depending on the various parameters present in the problem, this vortex may exhibit various types of behaviour, including pure rotation (as in Kirchhoff's case), periodic shape pulsations, infinite stretching and remaining absolutely fixed in space.

For an arbitrary two-dimensional compact vorticity distribution a general approach based on Clebsch variables was formulated by Hill (1884). As an example of the possibilities of this line of approach, the solution for the Kirchhoff vortex was obtained. Even after more than a century this method seems to deserve further extension and application to two-dimensional vortex problems; to our knowledge, this approach has not received any attention since that time.

A considerable simplification in the analysis of two-dimensional vortex flows can be obtained by considering the case of *steady* motion. For such motions, it was shown by Stokes (1842) that any flow field represented by the stream function $\psi(x, y)$, defined as

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \tag{1.1}$$

with (u, v) the velocity components in the Cartesian coordinate system (x, y), is a solution of the two-dimensional Euler equations for incompressible fluid provided that it satisfies the equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = f(\psi), \tag{1.2}$$

where $f(\psi)$ is an arbitrary function of ψ.

The first person to try to solve (1.2) for a non-uniform vorticity distribution was Lamb. In the second edition of his treatise *Hydrodynamics* (1895) he considered a possible solution for the case of a linear relationship $f(\psi)$:

> Again, if we put $f(\psi) = -k^2\psi$. where k is a constant, and transform to polar coordinates r, θ, we get
>
> $$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \frac{1}{r^2}\frac{d^2\psi}{d\theta^2} + k^2\psi = 0 \qquad (ii),$$
>
> which is satisfied by
>
> $$\psi = CJ_s(kr) \left.\begin{matrix}\cos \\ \sin\end{matrix}\right\} s\theta \qquad (iii),$$
>
> where J_s is a 'Bessel's Function.' This gives various solutions consistent with a fixed circular boundary of radius a, the admissible values of k being determined by
>
> $$J_s(ka) = 0. \qquad (iv).$$
>
> The character of these solutions will be understood from the properties of Bessel's Functions, of which some indication will be given in Chapter VIII.

The description of this solution was rather sketchy, and details were not given. The same possible solution was hinted at in the textbook by Wien (1900) and in the review paper by Love (1901). Both authors devoted only a short sentence to this problem, without any elaborations about the choice of the particular ψ solution given by (iii) above. In the third edition of *Hydrodynamics* (1906), however, Lamb added to the quotation from 1895 given above (with changes in the numbering of the equations and now omitting the quotes when referring to the Bessel function) a significant statement:

> Suppose, for example, that in an unlimited mass of fluid the stream-function is
>
> $$\psi = CJ_1(kr)\sin\theta, \tag{10}$$
>
> within the circle $r = a$, whilst outside this circle we have
>
> $$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta. \tag{11}$$
>
> These two values of ψ agree for $r = a$, provided $J_1(ka) = 0$. Moreover, the tangential velocity at this circle will be continuous, provided the two values of $\partial\psi/\partial r$ are equal, *i.e.* if
>
> $$C = \frac{2U}{kJ_1'(ka)} = -\frac{2U}{kJ_0(ka)}. \tag{12}$$
>
> If we now impress on everything a velocity U parallel to Ox, we get a species of cylindrical vortex travelling with velocity U through a liquid which is at rest at infinity. The smallest of the possible values of k is given by $ka/\pi = 1.2197$; the relative stream-lines *inside* the vortex are then given by the diagram on p. 272, provided the dotted circle be taken as the boundary ($r = a$). It is easily proved, by Art. 156 (1), that the 'impulse' of the vortex is represented by $2\pi\rho a^2 U$.

This dipolar solution with a continuous vorticity distribution on a circular region is now generally referred to as the Lamb dipole (and in some cases also as the Batchelor dipole, with reference to the description of this vortex structure in Section 7.3 of Batchelor 1967). In quite a number of recent laboratory studies on two-dimensional coherent vortex structures, dipolar vortices were observed under different conditions that show strong resemblance to this theoretical dipole model (see e.g. Couder & Basdevant 1986; Nguyen Duc & Sommeria 1988; van Heijst & Flór 1989*a*, *b*; Voropayev *et al.* 1991). Moreover, Flierl *et al.* (1983) performed an experimental study and presented a theoretical description of a non-symmetrical circular dipole (the vortex was called 'modon' in that paper) with its centre moving steadily along a circular trajectory. A similar solution was also derived by Bliss in 1970 (see Saffman 1992, Section 9.6).

In summary, a number of exact solutions of the two-dimensional Euler equations describing compact vortex structures are now known, some of them being classical (Rankine circular vortex, Kirchhoff elliptical vortex, Lamb dipole), others being formulated relatively recently (Moore & Saffman and Kida vortices, Flierl–Stern–Whitehead dipole).

A thorough study of the Russian fluid dynamics literature published during the last century and at the beginning of this century has revealed that quite a few important contributions to vortex dynamics problems have been written (in Russian) which seem to have escaped the attention of the international fluid dynamics community. For example, important work on two-dimensional point vortex motions was carried out by N. E. Joukowskii (1847–1921) and his pupil D. N. Goryachev (1867–1949); their works went unnoticed for a long time, but were recently discussed in some detail by Aref, Rott & Thomann (1992). Other important contributions to two-dimensional vortex flows were made by the Russian scientist S. A. Chaplygin, a pupil and later a close associate of Joukowskii. At the turn of the last century he wrote two remarkable papers (Chaplygin 1899, 1903) in which he gave detailed descriptions of some of the exact vortex solutions referred to before. In particular, Chaplygin discussed the dipole solution (including non-symmetrical ones) with continuous vorticity distribution according to a linear relationship $f(\psi)$, and he also described an exact solution of an elliptical uniform vorticity patch in an external flow containing pure shear. Because Chaplygin's contributions seem to have escaped the attention of the fluid dynamicists working on two-dimensional vortex problems (some of his results were independently discovered much later by other investigators), his work deserves being described appropriately.

In the present paper we will give a brief account of Chaplygin's results as presented in his papers published in 1899 and 1903. In addition, some important aspects of Chaplygin's vortex solutions will be analysed in more detail. In § 2, the elliptical vortex in an ambient shear field will be discussed. Chaplygin's dipole solutions will be addressed in § 3, and an account of his non-symmetric dipolar vortex moving along a circular trajectory is given in § 4. Some conclusions and a concise discussion of Chaplygin's work are presented in § 5. Finally, a short account of Chaplygin's life and scientific career is presented in the Appendix.

2. The elliptical vortex in a shear flow: Chaplygin (1899); Moore & Saffman (1971); Kida (1981)

In the introduction to his paper Chaplygin (1899) stated the following problem (translated from Russian in the specific terminology of that time):

> In this paper we consider the following case of motion in an unbounded mass of fluid: all motion is parallel to the coordinate plane OXY; the velocity components u and v are continuous in the entire flow domain; all the fluid contains vortices; vortex lines are parallel to the OZ axis; the angular velocity Ω of vortex rotation inside an elliptical cylinder with OZ axis is constant and equals to $A+\omega$, and in the rest of the fluid $\Omega = A$; the velocity infinitely far from OZ is parallel to OX and $u = -2Ay$. We will show that the inner vortex cylinder will change its form according to a certain law, rotating with a variable angular velocity around the OZ-axis.†

In order to solve this problem an assumption about the inner vorticity region was made. It was assumed that the initial elliptical vortex cylinder will always remain elliptical (figure 1 a) with the same value of vorticity, but both its principal axes a and b and the orientation angle ϕ with the fixed OX-axis depend on time. This supposition was verified during the course of the solution.

The problem of finding the stream function ψ such that the velocity components are defined according (1.1) for the whole two-dimensional motion under a known instant vorticity distribution 2Ω was reduced to the Poisson equation

$$2\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right). \tag{2.1}$$

That equation was solved by means of a traditional approach for that time (Kirchhoff 1876), namely, by searching the attraction potential of an elliptical cylinder

$$F(x', y') \equiv \frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1 = 0, \tag{2.2}$$

† It should be noted that here the term 'angular velocity' means $\omega_z/2$, with ω_z the z-component of the vorticity vector as used in modern notation.

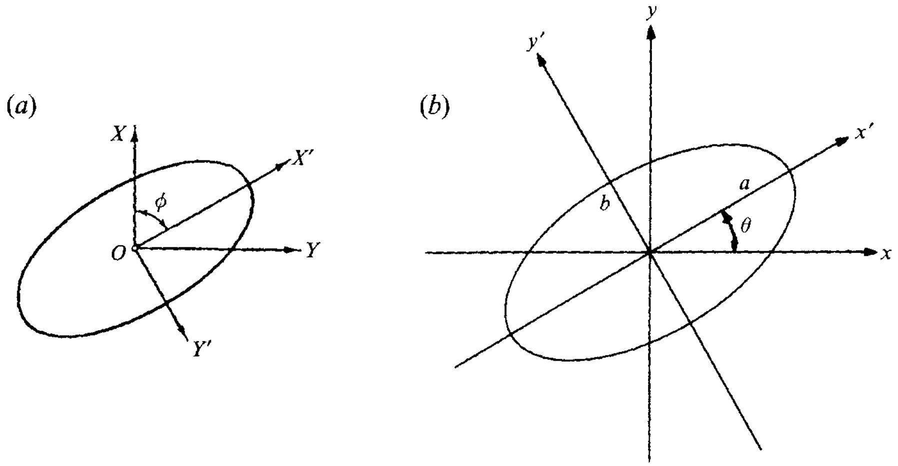

FIGURE 1. Schematic representation of the elliptical vortex and definition of the coordinate axes as used by (*a*) Chaplygin (1899) and (*b*) Kida (1981).

where

$$x' = x\cos\phi + y\sin\phi, \quad y' = -x\sin\phi + y\cos\phi, \tag{2.3a}$$

$$x = x'\cos\phi - y'\sin\phi, \quad y = x'\sin\phi + y'\cos\phi. \tag{2.3b}$$

After some algebra and by satisfying the conditions of shear flow at infinity, Chaplygin obtained the following expressions for the stream function inside (index i) and outside (index e) the elliptical cylinder:

$$\psi_e = -Ay^2 - \omega ab\ln(\alpha+\beta) - \frac{\omega ab}{\alpha+\beta}\left(\frac{x'^2}{\alpha}+\frac{y'^2}{\beta}\right), \tag{2.4a}$$

$$\psi_i = -Ay^2 - \omega ab\ln(a+b) - \frac{\omega ab}{a+b}\left(\frac{x'^2}{a}+\frac{y'^2}{b}\right), \tag{2.4b}$$

where

$$\alpha = (a^2+\lambda)^{1/2}, \quad \beta = (b^2+\lambda)^{1/2} \tag{2.5}$$

and λ is the non-negative root of the equation

$$\frac{x'^2}{a^2+\lambda}+\frac{y'^2}{b^2+\lambda} = 1. \tag{2.6}$$

Then the velocity field inside and outside the elliptical cylinder is given by

$$u_e = -2Ay - \frac{2\omega ab}{\alpha+\beta}\left(\frac{x'\sin\phi}{\alpha}+\frac{y'\cos\phi}{\beta}\right), \tag{2.7a}$$

$$v_e = \frac{2\omega ab}{\alpha+\beta}\left(\frac{x'\cos\phi}{\alpha}-\frac{y'\sin\phi}{\beta}\right), \tag{2.7b}$$

$$u_i = -2Ay - \frac{2\omega ab}{a+b}\left(\frac{x'\sin\phi}{a}+\frac{y'\cos\phi}{b}\right), \tag{2.8a}$$

$$v_i = \frac{2\omega ab}{a+b}\left(\frac{x'\cos\phi}{a}-\frac{y'\sin\phi}{b}\right). \tag{2.8b}$$

At the boundary of the elliptical cylinder (where $\lambda = 0$, $\alpha = a$, $\beta = b$) equations (2.7) and (2.8) yield $u_i = u_e$ and $v_i = v_e$, implying that the velocity field is continuous.

In order to obtain the equations for $a(t), b(t)$ and $\phi(t)$ the kinematic condition on the boundary was used. That condition states that the elliptical boundary (2.2) is always composed of the same fluid particles. It means that

$$\frac{\mathrm{D}F}{\mathrm{D}t} \equiv \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0. \tag{2.9}$$

Conservation of the cross-sectional area (as a consequence of Helmholtz's 1858 vortex theorems) of the cylinder implies

$$a(t)b(t) = a_0 b_0, \tag{2.10}$$

with a_0 and b_0 the principal axes of the initial elliptical vortex. After some algebraic transformations (collecting the terms with mutual multipliers x'^2, y'^2, $x'y'$), Chaplygin obtained the following system of ordinary differential equations for a and ϕ:

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -2aA\sin\phi\cos\phi, \tag{2.11a}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = 2A\frac{a^2\sin^2\phi - b^2\cos^2\phi}{a^2 - b^2} + \frac{2\omega ab}{(a+b)^2}. \tag{2.11b}$$

When $A = 0$ it follows that

$$a(t) \equiv a_0, \quad b(t) \equiv b_0, \quad \frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{2\omega a_0 b_0}{(a_0+b_0)^2}. \tag{2.12}$$

This solution represents the Kirchhoff vortex (Kirchhoff 1876) – the uniformly rotating elliptical patch of fixed shape.

By introducing the eccentricity

$$z = \frac{b}{a}, \tag{2.13}$$

and using (2.10), equations (2.11 a, b) can be put in the more convenient form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 4Az\sin\phi\cos\phi, \tag{2.14a}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = 2A\frac{\sin^2\phi - z^2\cos^2\phi}{1 - z^2} + \frac{2\omega z}{(1+z)^2}. \tag{2.14b}$$

Chaplygin showed that this nonlinear system (2.14) has an integrable multiple $Z(z) = (z^2-1)/z^2$ and that the following first integral exists:

$$A\left(\frac{\sin^2\phi}{z} + z\cos^2\phi\right) = \omega\ln\frac{Kz}{(1+z)^2}. \tag{2.15}$$

Here K is an integration constant which is determined by the initial conditions

$$z(0) = z_0 = \frac{b_0}{a_0}, \quad \phi(0) = \phi_0. \tag{2.16}$$

According to (2.15)

$$m\ln\frac{K}{4} = \frac{\sin^2\phi_0}{z_0} + z_0\cos^2\phi_0 - m\ln\frac{4z_0}{(1+z_0)^2}, \tag{2.17}$$

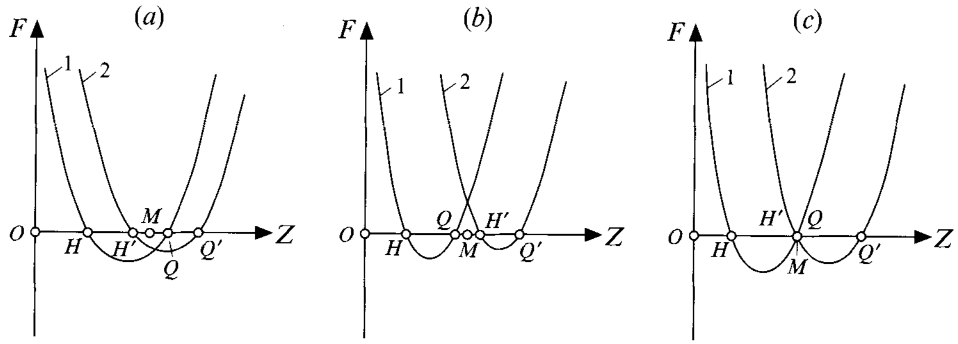

FIGURE 2. Schematic representation of the possible types of behaviour of the eccentricity $z(t)$ according to (2.20) with $m > 0$ for the cases (*a*) $m \ln K/4 > 1$, (*b*) $m \ln K/4 < 1$ and (*c*) $m \ln K/4 = 1$. The pictures are reproduced from Chaplygin (1899), figures 8, 9 and 10.

with $m = \omega/A$. The function $4z/(1+z)^2$ is always less than or equal to 1, and, therefore, for $m > 0$ the value $m \ln K/4$ is positive.

From (2.15) one derives

$$\sin^2\phi = -\frac{z}{1-z^2}F(z), \qquad \cos^2\phi = \frac{z}{1-z^2}F\left(\frac{1}{z}\right), \tag{2.18}$$

where

$$F(z) = z - m\,\ln\frac{K}{4} - m\,\ln\frac{4z}{(1+z)^2}. \tag{2.19}$$

Therefore the variable ϕ can be excluded from system (2.14) and the equation for $z(t)$ has the form

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{4Az^2}{(z^2-1)}\left\{-F(z)F\left(\frac{1}{z}\right)\right\}^{1/2}, \tag{2.20}$$

The solution $z(t)$ of this equation with initial condition $z(0) = z_0 < 1$ (for definition purposes, we take $b_0 < a_0$ at the initial moment) gives us through (2.10), (2.13) and (2.15) the instant form (axes a and b) and the orientation (the angle ϕ of the major axis a with the X-axis) of the elliptical patch. Then using (2.7) and (2.8) it is possible to obtain the velocity field in the entire flow domain.

Chaplygin showed that for $m > 0$ three different types of motion exist, depending on the value of $m \ln K/4$. These cases are graphically illustrated in figure 2(*a–c*) (they correspond to figures 8 – 10 of the original paper). Here the curves labelled 1 and 2 represent the functions $F(z)$ and $F(1/z)$, and $OM = 1$. The points represent the roots of the equation $F(z) = 0$ and Chaplygin traced their behaviour graphically for various values of the parameter $m \ln K/4$. The points H, Q and H', Q' represent the zeros of these functions, H' and Q' are inverse to Q and H with respect to the point M.

For the case $m \ln K/4 > 1$ the value of $z(t)$ is changed periodically between the points H and H' (figure 2*a*). Then according to (2.18) both major and minor axes of the ellipse will in turn coincide with the X-axis. Thus, in that case the elliptical patch *rotates* counterclockwise with non-uniform angular velocity.

For the case $m \ln K/4 < 1$ the value of $z(t)$ is also changed periodically, but now between the points H and Q (figure 2*b*). Then, according to (2.18), $\cos^2\phi > 0$ and, therefore, the minor axis of the ellipse never coincides with the X-axis. The major axis of the elliptical patch *oscillates* around the X-axis. For that case Chaplygin also noted (without any further comments) the remarkable possibility of a *steady* solution, when points H and Q coincide and curve 1 only touches the z-axis. This special case can be obtained for any value of m, taking

$$z_0 = z_{min} = \frac{[(m+1)^2 + 4m]^{1/2} - m - 1}{2} < 1, \quad \phi_0 = 0, \tag{2.21}$$

and, consequently, $F(z_{min}) = 0$. In that case the elliptical cylinder has a fixed shape ($z = z_{min}$) and a fixed orientation ($\phi = 0$), and the whole fluid motion is stationary. It is easy to understand that this steady elliptical patch is stable with respect to small changes in z_0 or ϕ_0 in (2.21): both of them shift curve 1 downward, and the oscillating solution is thus obtained.

In the case $m \ln K/4 = 1$ (figure 2*c*) the cross-section of the elliptical vortex cylinder changes continuously from an ellipse to a circle and vice versa. Once the cross-section has become circular (with $z = 1$), equation (2.20) reduces in the limit $z \to 1$ to

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -2A,$$

indicating that the shape again has to change into an ellipse. This process is repeated permanently.

The case $m < 0$ can be analysed in the same manner. We found that for $m \ln K/4 > 1$ the motion is oscillatory, with the major axis coinciding periodically with the Y-axis. For $m \ln K/4 < 1$ (positive or negative) the elliptical patch will perform a rotational motion in a clockwise direction. The case $m \ln K/4 = 1$ corresponds with the periodic changes from an ellipse to a circle.

The remaining part of Chaplygin's paper is devoted to the determination of the pressure field $p(x, y)$ in the whole fluid — a question which is usually omitted now when working in terms of ω–ψ functions. Starting from the Euler equations in the following form:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\frac{u^2 + v^2}{2} + 2v\Omega, \tag{2.22a}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{\partial v}{\partial t} - \frac{\partial}{\partial y}\frac{u^2 + v^2}{2} - 2u\Omega, \tag{2.22b}$$

where ρ is the constant fluid density, he obtained the Cauchy–Lagrange integral

$$\frac{p}{\rho} + \frac{u^2 + v^2}{2} + C = -2\Omega\psi - \int \left(\frac{\partial u}{\partial t}\,\mathrm{d}x + \frac{\partial v}{\partial t}\,\mathrm{d}y\right) \tag{2.23}$$

with the unknown C depending only on time.

After tremendous transformations occupying several printed pages, Chaplygin obtained expressions for the pressure inside and outside the elliptical cylinder. In a form slightly different from the original paper, these expressions are

$$\begin{aligned}\frac{p_i - p_0}{2\rho} &= A\omega\frac{ab}{(a+b)^2}\,(x'^2 + y'^2) + A\omega\frac{2}{(a+b)^2}\,(bx'\cos\phi - ay'\sin\phi)^2 \\ &\quad + \omega^2\frac{2ab}{(a+b)^3}(bx'^2 + ay'^2),\end{aligned} \tag{2.24}$$

$$\frac{p_e - p_0}{2\rho} = A\omega ab \ln\frac{\alpha+\beta}{a+b} + \omega^2\frac{a^2b^2}{(a+b)^2} - \omega^2\frac{a^2b^2}{(\alpha+\beta)^2} - A\omega\frac{ab}{\alpha\beta}\frac{\lambda x'y'}{(\alpha+\beta)^2}\sin 2\phi$$

$$+A\omega\frac{ab}{a^2-b^2}\left[\frac{2\lambda}{\alpha+\beta}\left(\frac{x'^2}{\alpha}+\frac{y'^2}{\beta}\right)-\alpha\beta+ab\right]\cos 2\phi$$
$$+A\omega\frac{ab}{(\alpha+\beta)^2}(x'^2+y'^2)+A\omega\frac{2ab}{\alpha\beta(\alpha+\beta)^2}(\beta x'\cos\phi-\alpha y'\sin\phi)^2$$
$$+\omega^2\frac{2a^2b^2}{\alpha\beta(\alpha+\beta)(a+b)^2}(\beta x'^2+\alpha y'^2), \tag{2.25}$$

where p_0 is the pressure at the origin. Again, when $\lambda=0$, $\alpha=a$, $\beta=b$ we have continuity of the pressure across the boundary.

The first term in (2.25) becomes infinite for $x', y' \to \infty$. This implies a most remarkable result: for ω and A of the same sign the pressure at infinity has an infinite value; when ω and A are of opposite sign the pressure p_0 in the centre of the elliptical cylinder has to be very large (strictly speaking, infinite) in order to prevent a negative pressure at infinity. The reason for such a behaviour can be understood from equation (2.4*a*) for the stream function ψ_e. According to (2.23) the term $-Ay^2$ will be cancelled by the term $-2Ay$ in u_e, but the second term $\omega ab\ln(\alpha+\beta)$ gives an infinite contribution to the external pressure. Only for the cases $\omega=0$ (pure shear flow) or $A=0$ (Kirchhoff vortex) is the pressure finite in the entire flow domain.

Although Chaplygin's 1899 paper was mentioned in the review papers by Love (1901, p. 123) and Auerbach (1908, p. 1061) – his analysis referred to as a generalization of the Kirchhoff vortex – it seems to have escaped the attention of the later fluid dynamics community. In 1971 Moore and Saffman turned independently to the same problem. They considered (Moore & Saffman 1971) the possibility of a steady solution for an elliptical patch of uniform vorticity ω_0 in an external flow field for the cases of both irrotational strain and simple shear. They presented the expression for ψ_e in elliptical coordinates and also investigated in much more detail the dependence of that steady solution on the parameters of strain (e/ω_0) and shear (γ/ω_0). In 1981 Kida generalized (Kida 1981) that problem to the unsteady motion of an elliptical patch of uniform vorticity distribution ω_K in an ambient flow field containing both shear and strain (figure 1*b*). That is, the velocity components of the background flow are

$$u_\infty(x,y)=ex-\gamma y,\quad v_\infty(x,y)=-ey+\gamma x, \tag{2.26}$$

where e and γ are the strain and shear at infinity, respectively. The governing equations for the length $a(t)$ of the major axis of the rotating ellipse and its orientation angle $\theta(t)$ to the OX-axis are

$$\frac{\mathrm{d}a}{\mathrm{d}t}=ea\cos 2\theta, \tag{2.27a}$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-e\frac{a^2+b^2}{a^2-b^2}\sin 2\theta+\omega_K\frac{ab}{(a+b)^2}+\gamma. \tag{2.27b}$$

Here $b(t)$ is the minor axis of the ellipse connected with $a(t)$ by means of (2.10).

The set of equations (2.27) contains three arbitrary parameters ω_K, e, γ and possesses very interesting types of behaviour, as studied by Kida (1981) and in some following papers (see e.g. Dritschel 1990; Polvani & Wisdom 1990; Dhanak & Marshall 1993).

A very important, but at the same time very difficult, question concerns the stability of the elliptical vortex. Following Love's (1893) analysis of the stability of the Kirchhoff vortex, Moore & Saffman (1971) performed a study of the linear stability of a steady vortex patch in an external velocity field for the cases of both pure strain and pure shear. The more general case of the nonlinear stability of a non-stationary elliptic vortex, but only for the case of pure strain flow, was studied by Dritschel (1990).

For the specific case $e = A$, $\gamma = -A$ (that is a pure shear flow in the directions inclined at $\pi/4$ to the x-axis), by putting

$$\theta = \tfrac{3}{4}\pi - \phi, \quad \omega_K = -2\omega, \tag{2.28}$$

in (2.27) one obtains exactly Chaplygin's set of equations (2.11) with two arbitrary parameters ω and A. Therefore, Chaplygin's solution represents one specific case of the more general Kida solution, namely that of an elliptical vortex patch in a simple Couette shear flow.

3. Dipolar vortex moving along a straight line: Chaplygin (1903); Lamb (1895,1906)

In 1903 Chaplygin published another remarkable paper (Chaplygin 1903) devoted to the motion associated with a compact vorticity distribution in a two-dimensional unbounded inviscid flow. In the introduction of that paper he gave a precise formulation of the problem:

> Consider an unbounded mass of incompressible fluid in which the motion is parallel to the OXY plane; let the motion outside some circular cylinder be irrotational, the velocity being equal to zero at infinity. The question is to find a distribution of vortex lines inside the cylinder that gives rise to a uniformly translating vortex column with a continuous velocity distribution and with a positive pressure all around.

As a first example of the solution Chaplygin considered in detail a case of rectilinear motion of a circular vortex of radius a with a constant translation velocity v_0. By superimposing on the whole fluid a uniform velocity $-v_0$ he obtained a stationary problem of a steady vortex cylinder placed in a potential flow with uniform velocity at infinity. By choosing the polar coordinate system (r, θ), with the origin at the centre of the cylinder, the stream function ψ_1 for the potential flow around the cylinder is written as

$$\psi_1 = v_0 \left(r - \frac{a^2}{r} \right) \sin\theta, \quad r > a. \tag{3.1}$$

Inside the vortex cylinder the stream function ψ has to satisfy equation (1.2), which in polar coordinates is

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} = f(\psi). \tag{3.2}$$

Without any reference to Lamb (1895) or Wien (1900), Chaplygin choose $f(\psi)$ as a linear function

$$f(\psi) = -n^2\psi, \quad \omega = n^2\psi, \tag{3.3}$$

where n is a constant, and sought for a solution of (3.2) of the form

$$\psi(r, \theta) = Z(r)\sin\theta. \tag{3.4}$$

After satisfying the conditions for continuity of the velocities

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r} \tag{3.5}$$

at the boundary $r = a$, that is

$$\psi = \psi_1, \quad -\frac{\partial \psi}{\partial r} = -\frac{\partial \psi_1}{\partial r} \quad \text{at } r = a, \tag{3.6}$$

and after some algebra Chaplygin obtained the following solution:

$$\psi(r,\theta) = \frac{2v_0 a}{bJ_1'(b)} J_1\left(\frac{br}{a}\right)\sin\theta, \quad r \leqslant a, \tag{3.7}$$

where $b = 3.8317$ is the smallest positive root of the equation $J_1(b) = 0$, and $n = b/a$. Here we use the modern notation J_1, instead of his I_1, for the ordinary Bessel function.

Equations (3.1) and (3.7) present a complete solution for the steady problem. Chaplygin also plotted the pattern of stationary streamlines, which is here reproduced in figure 3(a) (this is figure 1 in the original paper). The velocity inside the vortex has a maximum value in the origin equal to $2.48v_0$. By superimposing on the solution (3.1), (3.7) a uniform flow of velocity v_0 in the negative x-direction (that is, take $\psi_0(r,\theta) = -v_0 r\sin\theta$) one obtains the solution of a circular vortex dipole travelling with constant velocity v_0 in the negative x-direction through a fluid, which is otherwise at rest at infinity.

As indicated in § 1, this solution is identical to that outlined by Lamb (1895) and described by Lamb (1906), see the quotations given in that Section. Because no references were made to each other's work, it is assumed that Chaplygin and Lamb arrived independently at the same vortex dipole solution, which is now generally referred to as the 'Lamb dipole'. According to the chronological order of the publications just mentioned, the name 'Chaplygin–Lamb dipole' may seem to be more appropriate.

Although Lamb only devoted a few sentences to the dipole solution, Chaplygin presented an extensive analysis of the characteristics of this solution. He determined the positions O_+, O_- at which the vorticity distribution in the interior region $r \leqslant a$:

$$\omega(r,\theta) = \frac{2v_0 b}{aJ_1'(b)} J_1\left(\frac{br}{a}\right)\sin\theta, \tag{3.8}$$

takes a maximum and minimum value, respectively. These points are situated symmetrically at

$$\delta = \frac{c}{b}a = 0.48a, \quad \theta = \pm\frac{\pi}{2}, \tag{3.9}$$

where $c = 1.8412$ is the smallest positive root of the equation $J_1'(c) = 0$.

Equation (3.8) shows that the vortex dipole has two symmetrical parts of positive and negative vorticity distributions in the lower and the upper part of the cylinder, respectively (figure 3a). The intensities of the vorticity in each of the parts are

$$\Gamma_+ = \int_\pi^{2\pi}\int_0^a \omega(r,\theta) r \,\mathrm{d}\theta\,\mathrm{d}r = 6.83v_0 a, \quad \Gamma_- = \int_0^\pi\int_0^a \omega(r,\theta) r\,\mathrm{d}\theta\,\mathrm{d}r = -6.83v_0 a, \tag{3.10}$$

indicating that the net intensity of the symmetrical dipole is equal to zero.

Also, by using directly the two-dimensional Euler equations Chaplygin calculated the pressure $p(r,\theta)$ in the fluid and found that

$$p(r,\theta) = p_\infty + \rho\frac{v_0^2}{2} - \rho\frac{u_r^2 + u_\theta^2}{2} - \rho\begin{cases}\frac{1}{2}n^2\psi^2, & r \leqslant a\\ 0, & r > a,\end{cases} \tag{3.11}$$

where p_∞ is the pressure at infinity and ρ is the fluid density. Since $\psi = 0$ at the boundary $r = a$, equation (3.11) gives a continuous pressure distribution. The minimum pressure arises at the points of maximum ψ, that is at points O_+, O_-, given by (3.9):

$$p_{min} = p_\infty + \rho\frac{v_0^2}{2} - 2\rho v_0^2\frac{J_1^2(c)}{J_0^2(b)} = p_\infty - 3.69\rho v_0^2. \tag{3.12}$$

The physical constraint $p_{min} > 0$ provides a limitation on the velocity v_0 of the vortex dipole motion.

The symmetric character of the vortex dipole solution led Chaplygin to consider the problem of a 'semi-cylindrical' vortex moving along a solid wall (figure 3*b*) with free slip conditions on it. It appeared that the lower pressure between points A and B (where the velocities are equal to v_0 and the pressure equals p_∞) results in a net force of magnitude $R = 1.92\rho a v_0^2$ (per unit length perpendicular to the (r, θ)-plane). In the full (circular) vortex dipole, the lower pressure in the central part might explain the tendency of the structure to remain compact.

Furthermore, Chaplygin considered a second example – the natural generalization of the symmetrical vortex solution to the case with the interior vortex flow being asymmetric with respect to the OX-axis.

By taking the function $f(\psi)$ inside the circle $r \leqslant a$ as

$$f(\psi) = -n^2(\psi - \lambda), \quad \omega = n^2(\psi - \lambda), \tag{3.13}$$

with λ an arbitrary constant and $n = b/a$ as before, Chaplygin found the following solution to equation (3.2):

$$\psi(r,\theta) = \frac{2v_0 a}{bJ_0(b)}J_1\left(\frac{br}{a}\right)\sin\theta + \lambda\left[1 - \frac{J_0\left(br/a\right)}{J_0(b)}\right], \quad r \leqslant a \tag{3.14}$$

(here the property $J_1'(b) = J_0(b)$ was used). Outside the circular region the stream function ψ_1 is again given by equation (3.1), representing the potential flow around a rigid cylinder. It is easy to verify that for *any* λ the conditions (3.6) are satisfied identically. Equations (3.1) and (3.14) thus represent a steady-state solution to the two-dimensional Euler equations. Chaplygin's original plot of the streamline pattern for some $\lambda > 0$ is reproduced here in figure 3(*c*). In the rest of the paper, this solution is referred to as the 'Chaplygin dipole'.

To this point we have only presented a review of the analysis given in Chaplygin's (1903) paper. However, the dipole solution (3.14) has some interesting characteristics which will now be considered in more detail.

From the expressions (3.1) and (3.14) for the stream function in the exterior and interior domains, respectively, we can derive the corresponding velocity and vorticity fields inside (index i),

$$u_r^{(i)}(r,\theta) = \frac{2v_0}{bJ_0(b)}\frac{a}{r}J_1\left(\frac{br}{a}\right)\cos\theta, \tag{3.15a}$$

$$u_\theta^{(i)}(r,\theta) = -\frac{2v_0}{bJ_0(b)}\left\{\left[J_0\left(\frac{br}{a}\right) - \frac{a}{br}J_1\left(\frac{br}{a}\right)\right]\sin\theta + \frac{b}{2}\frac{\lambda}{av_0}J_1\left(\frac{br}{a}\right)\right\}, \tag{3.15b}$$

$$\omega^{(i)}(r,\theta) = \frac{2v_0 b}{aJ_0(b)}\left[J_1\left(\frac{br}{a}\right)\sin\theta - \frac{b}{2}\frac{\lambda}{av_0}J_0\left(\frac{br}{a}\right)\right], \tag{3.16}$$

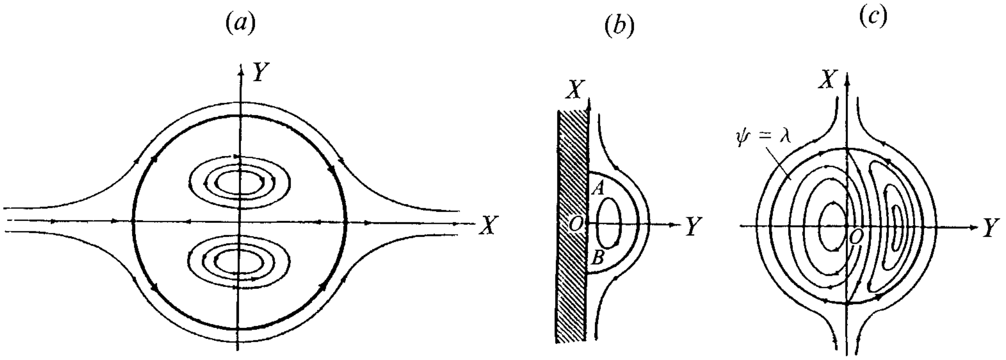

FIGURE 3. Streamline patterns of the dipole solution according to the Chaplygin solution: (a) the symmetrical dipole in an unbounded fluid; (b) the semi-cylindrical vortex moving along a flat rigid wall with free-slip conditions; and (c) the non-symmetrical dipole. The pictures are reproduced from Chaplygin (1903), figures 1, 2 and 3.

and outside (index e),

$$u_r^{(e)}(r,\theta) = v_0\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \quad u_\theta^{(e)}(r,\theta) = -v_0\left(1 + \frac{a^2}{r^2}\right)\sin\theta, \quad \omega^{(e)}(r,\theta) = 0 \quad (3.17)$$

the circle $r = a$. These expressions show that at the boundary $r = a$ the velocity field is continuous, whereas the vorticity distribution contains a jump discontinuity:

$$\omega^{(i)}(a,\theta) = -\lambda b^2/a^2, \quad \omega^{(e)}(a,\theta) = 0.$$

Contour plots of ψ and ω are presented in figures 4 and 5, respectively, for different values of the dimensionless parameter λ/av_0. The line $\psi = \lambda$ (bold line in the plots) divides the circular region $r \leqslant a$ into two parts. Inside that curve the vorticity is positive and outside it is negative. It is seen that with increasing values of λ the structure of the ω-distribution gradually changes into that of a monopolar vortex with a nearly circular positive-vorticity core surrounded by a circular band of negative vorticity. This can also be seen (figure 6) in the cross-sectional distributions along the y-axis of the vorticity ω and the velocity $u = u_r\cos\theta - u_\theta\sin\theta$: the positive vorticity has an almost symmetrical distribution for $\lambda/av_0 = 2$.

The points O_+ and O_- of maximum and minimum vorticity (where according (3.5) and (3.13) the velocity is equal to zero) still lie on the y-axis ($\theta = \pm\pi/2$), but they are located non-symmetrically with respect to the origin. Using (3.15) it is seen that their radii $r_+(\lambda) = a\rho_+$ and $r_-(\lambda) = a\rho_-$ are given by

$$J_0(b\rho_+) - \left(\bar{\lambda} + \frac{1}{b\rho_+}\right)J_1(b\rho_+) = 0, \quad J_0(b\rho_-) + \left(\bar{\lambda} - \frac{1}{b\rho_-}\right)J_1(b\rho_-) = 0, \quad (3.18)$$

with

$$\bar{\lambda} = \frac{b}{2}\frac{\lambda}{av_0}. \quad (3.19)$$

The equation of the curve $L_0 : r = a\rho_0(\theta)$ inside the vortex dipole at which $\omega = 0$ is

$$J_1(b\rho_0)\sin\theta - \bar{\lambda}J_0(b\rho_0) = 0. \quad (3.20)$$

Equations (3.18) and (3.20) can be easily solved numerically providing all neccessary data for further analysis.

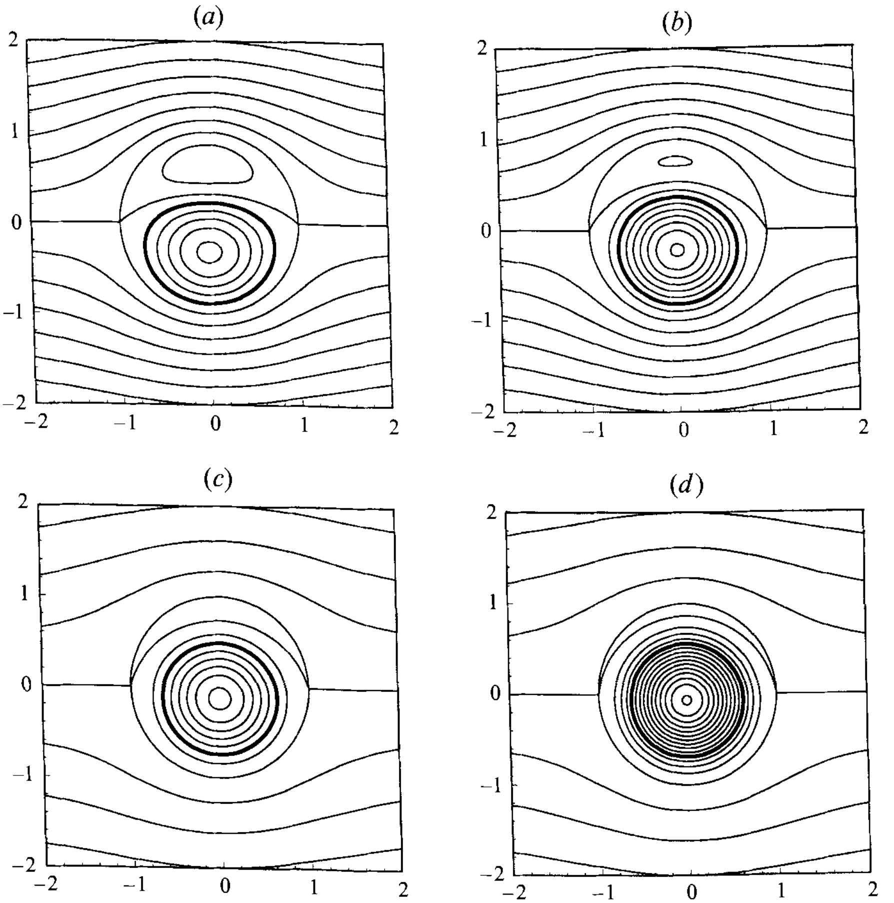

FIGURE 4. Streamline patterns of the non-symmetric Chaplygin dipole rperesented by (3.14) for $\lambda/av_0 = 0.25$ (*a*), 0.5 (*b*), 1.0 (*c*) and 2.0 (*d*). The bold line corresponds with $\psi = \lambda$. The streamlines are plotted for $\Delta\psi = 0.25\,av_0$ (a, b) and $\Delta\psi = 0.5\,av_0$ (c, d)

Integration of the vorticity distribution over the entire region $r \leqslant a$ shows that the net vorticity of the Chaplygin dipole is equal to zero. The positive value $\Gamma_+(\lambda)$ of the intensity on the region S_+ bounded by the curve L_0 (figure 7) is given by the integral

$$\Gamma_+ = \int\int_{S_+} \omega(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta. \tag{3.21}$$

It is in principle possible to calculate the surface integral (3.21) directly, by using the results of the numerical solution of equation (3.20), but this is a rather cumbersome procedure. A more simple way consists of using Stokes' theorem for (3.21) and putting the following:

$$\Gamma_+ = \oint_{L_0} (u_r\,\mathrm{d}r + u_\theta r\,\mathrm{d}\theta) = a\int_0^{2\pi} \left[u_r(a\rho_0(\theta),\theta)\rho_0'(\theta) + u_\theta(a\rho_0(\theta),\theta)\rho_0(\theta)\right]\,\mathrm{d}\theta, \tag{3.22}$$

where $\rho_0'(\theta)$ is easily obtained explicitly from (3.20), namely

$$\rho_0'(\theta) = -\frac{aJ_1(b\rho_0)\cos\theta}{\left[bJ_0(b\rho_0) - (1/\rho_0)J_1(b\rho_0)\right]\sin\theta + \bar{\lambda} bJ_1(b\rho_0)}. \tag{3.23}$$

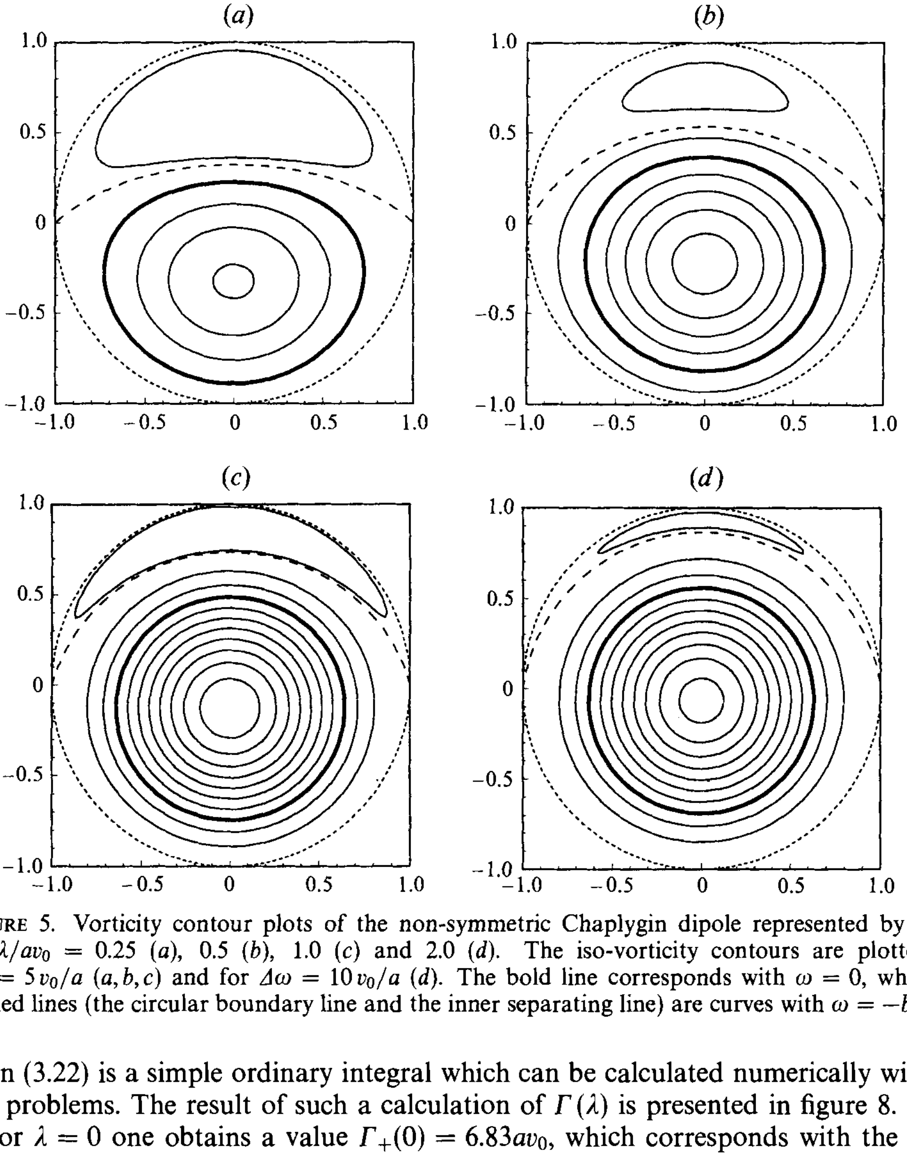

FIGURE 5. Vorticity contour plots of the non-symmetric Chaplygin dipole represented by (3.16) for $\lambda/av_0 = 0.25$ (*a*), 0.5 (*b*), 1.0 (*c*) and 2.0 (*d*). The iso-vorticity contours are plotted for $\Delta\omega = 5v_0/a$ (*a, b, c*) and for $\Delta\omega = 10v_0/a$ (*d*). The bold line corresponds with $\omega = 0$, while the dashed lines (the circular boundary line and the inner separating line) are curves with $\omega = -b^2\lambda/a^2$.

Then (3.22) is a simple ordinary integral which can be calculated numerically without any problems. The result of such a calculation of $\Gamma(\lambda)$ is presented in figure 8.

For $\lambda = 0$ one obtains a value $\Gamma_+(0) = 6.83av_0$, which corresponds with the value (3.10) of the symmetrical (Lamb) dipole. When $\lambda/av_0 > 1$ the value of $\Gamma_+(\lambda)$ increases practically linearly with λ. In order to understand this behaviour let us consider the asymptotical solutions of (3.18) and (3.20) for $\rho_+(\lambda)$ and $\rho_0(\theta)$ when $\bar{\lambda} \gg 1$. The radius $R(\theta)$ of the nearly circular domain (figure 7) with its centre at O_+ is easily found by means of the cosine theorem. After straightforward calculations one obtains

$$r_+(\lambda) \approx \frac{a}{b\bar{\lambda}}, \quad R(\theta) \approx \frac{ad}{b} + a\frac{\cos^2\theta}{2d\bar{\lambda}^2}, \quad \omega_{max} \approx -v_0\frac{2b}{aJ_0(b)}\bar{\lambda}\left(1 + \frac{3}{8\bar{\lambda}^2}\right), \tag{3.24}$$

where $d = 2.4048$ is the first zero of J_0. These relations show that for large $\bar{\lambda}$ the point O_+ of maximum positive vorticity tends to O, and that the shape of the $\omega > 0$ domain becomes almost circular, with an asymptotic radius $0.63a$. The data of figure 4(*d*) indicate that the vorticity distribution inside the $\omega > 0$ domain S_+ is practically independent of θ when $\lambda/av_0 > 1$. The total vorticity on S_+ can be approximated

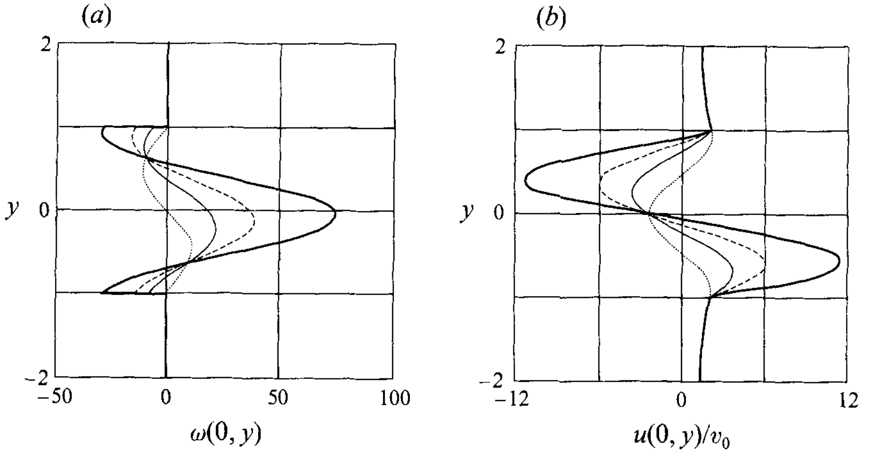

FIGURE 6. Cross-sectional distributions of: (a) the vorticity ω and (b) the velocity $u = u_r \cos\theta - u_\theta \sin\theta$ along the y-axis of the Chaplygin dipole for $\lambda/av_0 = 0$ (· · · ·), 0.5 (—), 1.0 (– – –) and 2.0 (▬).

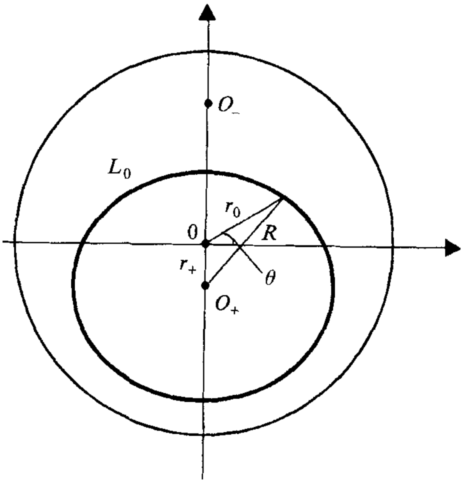

FIGURE 7. Schematic representation of the positive- and negative-vorticity regions of the non-symmetric Chaplygin dipole. The points O_+ and O_- denote the positions of maximum and minimum vorticity, respectively.

by calculating the volume of the semi-ellipsoid with a circular base S_+ and a major semi-axis ω_{max}. This yields

$$\frac{\Gamma_+}{av_0} \approx -\frac{2\pi d^2}{3J_0(b)}\frac{\lambda}{av_0} \tag{3.25}$$

and this dependence is shown by the dashed line in figure 8.

If again a uniform flow of velocity v_0 in the negative direction of the x-axis is superimposed on the solutions (3.1) and (3.14), we get the final solution for the Chaplygin dipole – the circular vortex travelling with uniform velocity v_0 in the negative direction along the x-axis through a fluid which is at rest at infinity. The

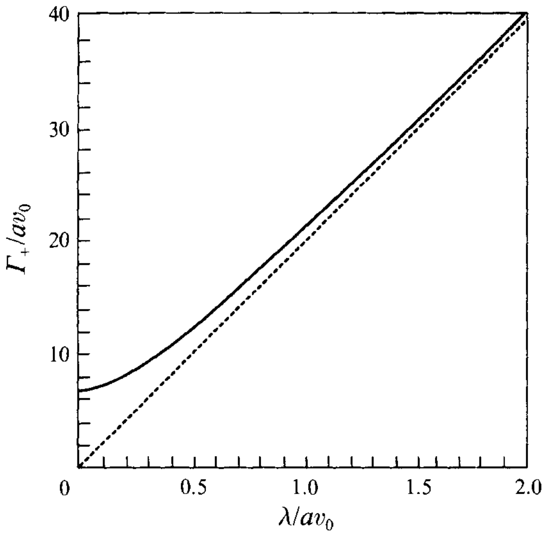

FIGURE 8. Graphical representation of the normalized positive intensity Γ_+/av_0 of the Chaplygin dipole as a function of the parameter λ/av_0: the solid line represents the exact calculation according to (3.22), whereas the dashed line denotes the approximation (3.25).

vorticity distribution $\omega(x, y, t)$ inside the circular domain is given by (3.16) putting $r^2 = (x + v_0 t)^2 + y^2$, $\sin\theta = y/r$.

Using the general formulae (Batchelor 1967, Section 7.3) for the components P, Q of the linear momentum, the angular momentum N about the origin, and the kinetic energy W of the fluid which is rest at infinity,

$$P = \rho \int_S y\omega \, \mathrm{d}S, \quad Q = -\rho \int_S x\omega \, \mathrm{d}S, \quad N = -\frac{\rho}{2} \int_S (x^2 + y^2)\omega \, \mathrm{d}S, \quad W = \frac{\rho}{2} \int_S \psi\omega \, \mathrm{d}S, \tag{3.26}$$

one obtains for the Chaplygin dipole

$$P = -2\pi a^2 \rho v_0, \quad Q = 0, \quad N = 2\pi a^3 \rho v_0 \frac{\lambda}{av_0}, \quad W = \pi a^2 \rho v_0^2 \left[1 + \frac{b^2}{2}\left(\frac{\lambda}{av_0}\right)^2\right]. \tag{3.27}$$

Obviously, the linear momentum (P, Q) does not depend on λ, and the values of P and Q are the same as for the symmetric Chaplygin–Lamb dipole ($\lambda = 0$).

An important invariant of two-dimensional inviscid motion is the enstrophy G, which is defined by

$$G = \frac{1}{2} \int_S \omega^2 \, \mathrm{d}S. \tag{3.28}$$

Using (3.16), the enstrophy of the Chaplygin dipole is found to be

$$G = \frac{b^2}{a^2} \frac{W}{\rho} = \pi b^2 v_0^2 \left[1 + \frac{b^2}{2}\left(\frac{\lambda}{av_0}\right)^2\right]. \tag{3.29}$$

Minimum values of the enstrophy G, and the kinetic energy W, are apparently obtained for the symmetric case $\lambda = 0$.

Again, the question of stability is one of the most important, but Chaplygin did not mention it. The condition for linear stability of an arbitrary solution of (1.2) was formulated by Maxwell in a draft manuscript (1855 *a*)† and also in a letter to William Thomson (1855 *b*). In the latter he wrote:

> I have been investigating fluid motion with reference to stability and I have got results when the motion is confined to the plane of *xy*. I do not know whether the method is new. It only applies to an incompressible fluid moving in a plane.
>
> ...
>
> Put‡ $\chi = \frac{d^2\psi}{dx^2} + \frac{d^2\psi}{dy^2}$.
> Hence
> (A)
> $\frac{d\chi}{dt} = 0$ or $\chi = f(\psi)$ is the condition of steady motion as is otherwise known.
> (B)
> $f'(\psi)$ or $\frac{d\chi}{d\psi}$ must be negative for stability.
> When $f'(\psi)$ is positive the motion is unstable.
> When $f'(\psi) = 0$, χ is constant or 0.
> When χ is constant I think equilibrium is neutral.
> When $\chi = 0$ the whole motion is determined by the motion at a limiting curve so that there can be no finite displacement.

It should be noted that Maxwell anticipated the analysis performed more than a century later by Arnol'd (1965) and Drazin & Howard (1966); this is another story, however, which deserves in our opinion a separate study.

It is easy to verify that both the Lamb and the Chaplygin dipoles satisfy the condition $f'(\psi) < 0$ and, therefore, both are stable according to Maxwell's criterion. However, numerical simulations (R. Verzicco and J. Voskamp, private communications) of two-dimensional flow with the initial vorticity distribution (3.16), both for viscid ($Re = 1000$) and practically inviscid flow, indicate that the Chaplygin dipole is essentially unstable: when $\lambda > 0$ it sheds some negative vorticity which is left in its wake and the vortex structure is slightly deflected in the direction of the positive vorticity. When $\lambda = 0$ (the symmetric Chaplygin–Lamb dipole) such a dipolar structure moves along a straight line while conserving its compact shape. The reason of this difference is not quite clear yet, and deserves being investigated further.

4. Dipolar vortex moving along a circular path: Chaplygin (1903); Flierl, Stern & Whitehead (1983)

In the last part of his remarkable paper Chaplygin (1903) briefly describes one additional case of a steadily moving circular vortex dipole. For that purpose, he added to the non-symmetric dipole a monopolar Rankine vortex: a circular patch of radius a with a uniform vorticity $-2\kappa/a^2$. For this additional monopolar vortex the expressions for the stream function ψ_2 and the vorticity ω_2 are

$$\psi_2 = -\kappa \ln \frac{r}{a}, \quad \omega_2 = 0 \quad \text{for } r > a \tag{4.1}$$

$$\psi_2 = \frac{\kappa}{2}\left(\frac{r^2}{a^2} - 1\right), \quad \omega_2 = -\frac{2\kappa}{a^2} \quad \text{for } r \leqslant a. \tag{4.2}$$

† Here Maxwell missed a minus sign in the derivation, so his final result has to be changed accordingly.

‡ Maxwell used the notations d/dx and d/dy for the partial derivaties with respect to the coordinates x and y, and d/dt for the full derivative with respect to time ('as we pass along the path of the particle').

It is easily verified that such an addition still satisfies the condition $J(\omega, \psi) = 0$ in the two-dimensional inviscid vorticity equation for the flow to be steady.

In order to obtain a stationary solution to the Euler equations Chaplygin used a coordinate system that rotates uniformly with angular velocity κ/a^2. This approach is equivalent to adding a rigid-body rotation at infinity. Expressed in a polar coordinate system with the origin at the centre of the vortex dipole, Chaplygin's solution reads

$$\psi_{Ch}(r,\theta) = \begin{cases} v_0 \left(r - \dfrac{a^2}{r}\right) \sin\theta + \kappa \ln \dfrac{r}{a} - \dfrac{\kappa}{2}\left(\dfrac{r^2}{a^2} - 1\right), & r > a \\ v_0 \dfrac{2a}{bJ_0(b)} J_1(br/a) \sin\theta + \lambda \left[1 - \dfrac{J_0(br/a)}{J_0(b)}\right], & r \leqslant a, \end{cases} \tag{4.3}$$

$$\omega_{Ch}(r,\theta) = \begin{cases} 2\kappa/a^2, & r > a \\ v_0 \dfrac{2b}{aJ_0(b)} J_1(br/a) \sin\theta - \lambda \dfrac{b^2 J_0(br/a)}{a^2 J_0(b)}, & r \leqslant a. \end{cases} \tag{4.4}$$

For prescribed values of v_0 and a, this solution depends on two dimensionless parameters, namely λ/av_0 and κ/av_0. Examples of steady streamlines according (4.3) for some values λ and κ are presented graphically in figure 9. The streamline pattern inside the region $r \leqslant a$ is only dependent on λ, whereas the streamline pattern outside the vortex dipole depends only on κ.

By applying the Euler equations in the uniformly rotating frame Chaplygin also calculated the pressure field in the entire flow domain and proved its continuity across the dipole boundary $r = a$.

The exterior solution for the vortex dipole is a special case of the general solution for a rigid cylinder of radius $r = a$ with circulation $\tilde{\kappa}$ moving with velocity U through an inviscid fluid that rotates uniformly at angular velocity $\omega_0/2$ (see Batchelor 1967, Section 7.4):

$$\psi_B(r,\theta) = -\frac{\omega_0 r^2}{4} - U\left(r - \frac{a^2}{r}\right)\sin\theta - \frac{\tilde{\kappa}}{2\pi}\ln\frac{r}{a}. \tag{4.5}$$

The link with the present problem is provided by

$$\omega_0 = \frac{2\kappa}{a^2}, \quad \tilde{\kappa} = -2\pi\kappa, \quad U = -v_0. \tag{4.6}$$

Finally, by superimposing on the solutions (4.3), (4.4) a uniform rotation with angular velocity κ/a^2 about a central point (r_c, θ_c), with

$$r_c = \frac{v_0 a^2}{|\kappa|}, \quad \theta_c = -\frac{\pi}{2}\frac{\kappa}{|\kappa|},$$

Chaplygin obtained an expression for the flow due to a vortex dipole of net vorticity $2\pi\kappa$ moving at uniform speed along a circular trajectory with central point (r_c, θ_c) through an irrotational fluid that is at rest at infinity.

For the steady solution (4.3), (4.4) the functional relation between ω and ψ is

$$\omega = \begin{cases} 2\kappa/a^2, & r > a \\ b^2(\psi - \lambda)/a^2, & r \leqslant a. \end{cases} \tag{4.7}$$

Equation (4.7) shows that on the dipole boundary $r = a$, where $\psi = 0$, the vorticity has a jump $-(b^2\lambda + 2\kappa)/a^2$. The ω–ψ relationship is plotted in figure 10, and it can be seen that the jump is due to an upward shift $2\kappa/a^2$ of the horizontal

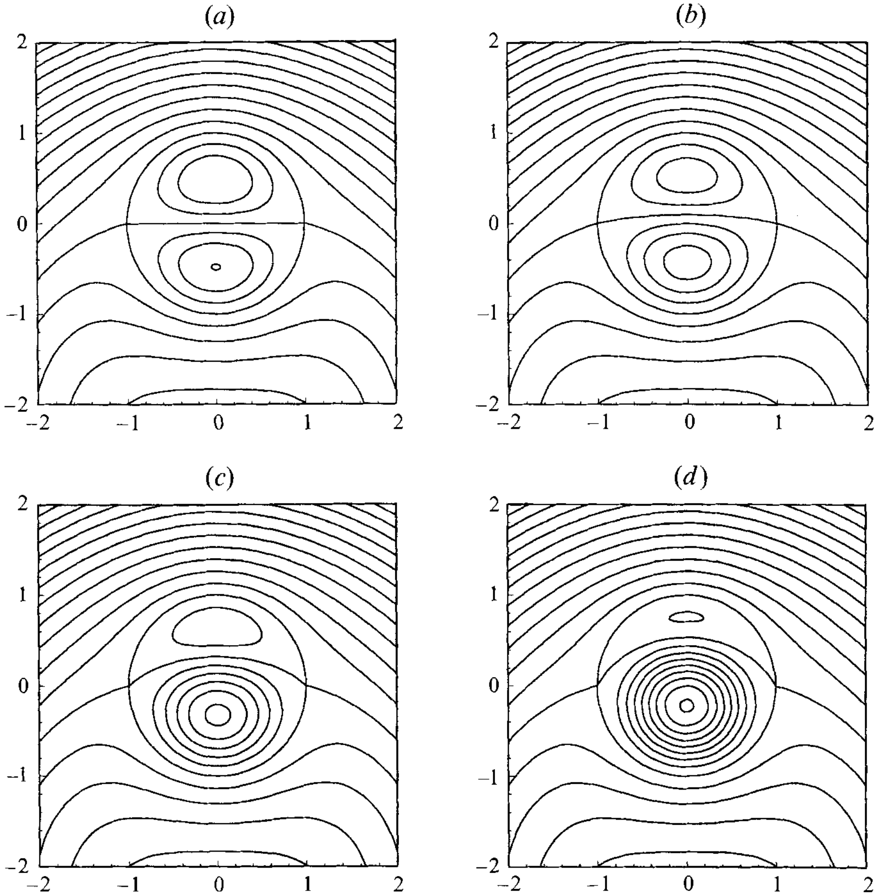

FIGURE 9. Streamline patterns of the Chaplygin dipole moving along a circular trajectory, as represented by (4.3). The streamlines are plotted for $\kappa/av_0 = -0.5$ and $\lambda/av_0 = 0$ (*a*), 0.06811 (*b*), 0.25 (*c*) and 0.5 (*d*). Case (*b*) corresponds to the FSW solution.

branch representing the exterior flow, and a downward shift $-b^2\lambda/a^2$ of the branch representing the interior of the dipole.

Chaplygin's solution was obtained independently a few years ago by Flierl *et al.* (1983) in a study of dipole formation by a jet in a uniformly rotating fluid. In the 'modon' frame (using their terminology and notation, and after correcting for some misprints) their steady solution (henceforth referred to as FSW) is given by

$$\psi_{FSW}(r,\theta) = \begin{cases} c\left(r - \dfrac{a^2}{r}\right)\sin\theta - c\epsilon\ln\dfrac{r}{a} + \epsilon\dfrac{r^2}{2a^2}, & r > a \\[2ex] \dfrac{2cJ_1(kr)}{kJ_0(ka)}\sin\theta + \dfrac{2c\epsilon}{k^2a}\left[1 - \dfrac{J_0(kr)}{J_0(ka)}\right], & r \leqslant a, \end{cases} \tag{4.8}$$

where c is the speed of the modon along the circular track, ϵ is the ratio of the modon radius a and the radius of its circular trajectory, and $ka = 3.8317$ is the first zero of the Bessel function J_1. Essentially the same solution (in slightly different notation) was formulated by Bliss in 1970 (see Saffman 1992, Section 9.6) and also

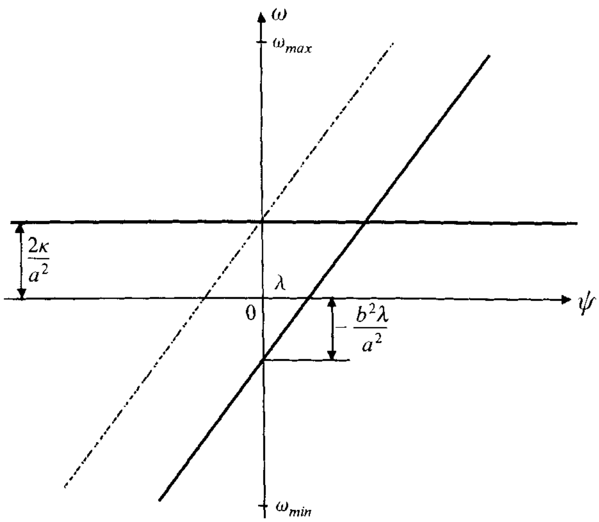

FIGURE 10. Graphical representation of the ω–ψ relationship of the Chaplygin dipole moving along a circular path, according to (4.7). The exterior field and the dipole's interior correspond to the horizontal and inclined lines, respectively. The solid lines denote the general case, while the dashed line represents the FSW case (4.10).

mentioned briefly by Nycander & Isichenko (1990). Experimentally such asymmetric vortex dipoles were also studied by Nguyen Duc & Sommeria (1988).

It is seen that for given c and a the solution (4.8) depends on a single dimensionless parameter, ϵ. This parameter defines the angular velocity $\Omega = c\epsilon/a$ and the radius of the track $A = a/\epsilon$. It is easy to prove that (4.8) is a particular but important case of Chaplygin's general solution (4.3). By putting in (4.3)

$$v_0 = c, \quad b = ka, \quad \lambda = \frac{2c\epsilon}{k^2 a}, \quad \kappa = -\epsilon c a \tag{4.9}$$

one obtains (4.8) (up to an additional constant $\epsilon/2$). The importance of the FSW solution lies in the fact that, in terms of Chaplygin's solution, we have

$$\lambda = -\frac{2\kappa}{b^2}, \tag{4.10}$$

so the jump in the vorticity at $r = a$ is *absent*. The ω–ψ relationship is again given by (4.7) but the interior branch now intersects the horizontal exterior branch exactly at the ω-axis. This FSW case is indicated by the broken line in figure 10.

5. Conclusions

Since the seminal paper of Helmholtz (1858) in which some fundamental vortex theorems were formulated, the subject of vorticity dynamics has continued to attract many fluid dynamicists. It is interesting to note that in this field, more than in any other branch of fluid mechanics, many independent discoveries were made both of the governing equations describing certain vortex flows and of some exact analytical solutions of these. One example is the careful analysis of the three-vortex problem by W. Gröbli in the dissertation published in 1877 (for a detailed historical survey, see Aref *et al.* 1992): quite a few of the results described in his thesis were rediscovered almost a century later! Another example is the localized induction approximation (LIA) for the motion of a vortex filament: the governing equations for this approximation had already been published in 1906 by the young Italian graduate student L.S. Da Rios, who worked under the supervision of T. Levi-Civita. As discussed by Ricca (1991), in the course of the present century these LIA equations have even been independently rediscovered even a few times.

The same appears to have occurred to two papers published by the Russian scientist S.A. Chaplygin at the turn of the last century. In these papers he analysed the flow due to an elliptical vortex patch in an ambient flow field with uniform vorticity (Chaplygin 1899) and the structure of a circular dipolar vortex with a continuous vorticity distribution according to a linear ω–ψ-relationship (Chaplygin 1903). Probably because these papers were written in Russian, they went unnoticed for a long time.

The analysis of the elliptical vortex patch presented in the 1899 paper was quoted by Love (1901) and Auerbach (1908) as a generalization of Kirchhoff's elliptical vortex. Independently, a generalization of the Kirchhoff vortex was made by Moore & Saffman (1971), who considered steady-state solutions for an elliptical vortex patch in pure straining and simple shear flows. After construction of such solutions (one of them was mentioned by Chaplygin only briefly) these authors go further and do the linear stability analysis for general disturbances of a vortex patch boundary. This does not appear to have been studied by Chaplygin at all. Kida (1981) published another generalization of Kirchhoff's unsteady vortex to the case of an ambient flow containing both strain and vorticity. Again, Chaplygin's solution was not mentioned, although it corresponds with a special case of the Kida solution, viz. that of an elliptical vortex patch in a simple Couette shear flow.

In the second paper, discussed in the previous sections, Chaplygin (1903) presents an analytical solution of a dipolar vortex with distributed vorticity inside a circular area, according to a linear ω–ψ relationship. Apparently, he was not aware of the texts of Lamb (1895) and Wien (1900), in which brief remarks were made about the *possibility* of having such solutions. Chaplygin presented analytical solutions both for the dipolar vortex moving steadily along a straight path and for the dipole moving steadily along a circular track. In the former case, his symmetrical dipole solution is identical to that outlined by Lamb (1895) and described in some more detail by Lamb (1906). As discussed in the preceding sections of the present paper, however, Chaplygin (1903) provides a much more detailed analysis of the dipole characteristics. Moreover, Chaplygin gave a generalization to the case of a non-symmetric dipole moving steadily along a straight line, and the details of this remarkable solution were discussed in § 3 of the present paper.

Another generalization concerned the addition of a monopolar 'rider' solution, leading to a non-symmetric dipolar vortex moving along a circular path is discussed in § 4. An analytical solution of such a dipolar vortex structure was also presented much later by Flierl *et al.* (1983), and again Chaplygin's earlier work appeared to be not known. The FSW solution is a special case of Chaplygin's general solution, viz. that in which the jump in the vorticity at the vortex edge is exactly zero (see figure 10).

The main purpose of the present paper is to bring the earlier work of Chaplygin on two-dimensional coherent vortex solutions to the attention of modern fluid dynamicists, since most of them appear to be ignorant about his important contributions to the subject. In contrast to the widespread ignorance of Chaplygin's early papers in the western parts of the world, his contributions are still known in some small circles of Russian scientists, as is apparent from the relatively recent studies of Slezkin (1988) and Yarmitskii (1992) in which a generalization of Chaplygin's cylindrical vortex is discussed. Such references were found to be very scarce though.

The authors gratefully acknowledge the help of Roberto Verzicco and Jan Voskamp, who showed by numerical simulation the instability of the non-symmetric ($\lambda > 0$) Chaplygin dipole. We are indebted to Professor H. Aref who brought the paper of Ricca (1991) to our attention. Also, we are thankful to Menno Eisenga for some helpful discussions about Maxwell's stability analysis.

This paper was written during a working visit of V. V. M. to Eindhoven, for which financial support by the Dutch Organization for Scientific Research (NWO) is gratefully acknowledged.

Appendix. Biographical sketch of Sergey Alekseevich Chaplygin (1869–1942)

S. A. Chaplygin was born on 5 April 1869 in Ranenburg, a town not far from Moscow. He studied at the Department of Physics and Mathematics of the Moscow University, where he attended the lectures on hydrodynamics by N. E. Joukowskii. In 1894 he became an assistant professor at Moscow University and a few years later he was appointed as a full professor. During a period of almost 20 years he held positions as a lecturer in mechanics and mathematics at various institutes of higher education in Moscow, including the Moscow University and the Moscow High School of Technology. In the period 1921 to 1931 he was director of the Central Aero-Hydrodynamic Institute (TsAGI), a research institute founded in 1918 by Joukowskii; following this he headed the Institute's general theoretical team. In 1942 (at the age of 73) he died in Novosibirsk, to which TsAGI was evacuated during World War II.

Chaplygin did much pioneering work on several topics in gasdynamics, hydrodynamics, aerodynamics and also on some topics in general mechanics and pure mathematics. After his death his collected papers were published in 1948–1950 in 4 volumes (I, Theoretical mechanics and mathematics; II, Hydromechanics and aerodynamics; III, Lectures and presentations on mathematics and mechanics; IV, Lecture courses on theoretical mechanics), and selections of his many papers on mechanics and mathematics were re-published in 1954 and 1976.

His significant scientific contributions to the development of mechanics are cited in numerous books and papers. In this respect we mention Loitsyanskii's (1966) textbook on hydrodynamics and the review paper by Grigoryan (1965), to which the interested reader is referred. It is interesting to note that none of these publications mentions Chaplygin's (1899, 1903) early papers on vortex dynamics that form the subject of the present study.

Without attempting to review Chaplygin's complete scientific oeuvre we will briefly mention some of his most important contributions to the fields of hydrodynamics, gasdynamics and aerodynamics. In his magister dissertation, which was published in 1897 in the form of two large articles, and for which he was awarded the gold medal of the Russian Academy of Sciences, Chaplygin considered some new integrable cases of motion of a solid body in an inviscid fluid and gave (in line with Joukowskii's tradition) a clear geometrical model for its movement. In spite of the fact that they were published almost a century ago, these papers deserve more attention. In 1904 Chaplygin published his famous doctoral dissertation entitled 'On gas jets' (the defence of this thesis took place at the Moscow University in the year 1902), by which he made a fundamental contribution to the development of the theory of gasdynamics. In this remarkable work he described the elegant technique of changing from the physical plane of two-dimensional flow to the 'hodograph plane' of flow velocities, for which the equations of motion become linear. In the same year Chaplygin published an annotated Russian translation of two fundamental papers by Helmholtz, viz. on vortex motion (Helmholtz 1858) and on discontinuous motion in an ideal fluid. In 1906 Chaplygin and Joukowskii published a rigorous solution in terms of so-called bipolar coordinates to the problem of two-dimensional flow of a viscous fluid between two eccentric rotating cylinders (now commonly referred to as the journal bearing flow). Chaplygin also contributed greatly to various aerodynamic problems. During the XIIth Congress of the Russian Society for Natural Sciences and Medicine (which was held in Moscow from 28 December 1909 to 5 January 1910), in a discussion of a report by Joukowskii devoted to determination of the lift force on a two-dimensional airofoil, Chaplygin postulated that the fluid must leave the sharp trailing edge of the airofoil smoothly. This so-called Chaplygin–Joukowskii hypothesis is also linked with the German scientist Kutta, whose work on the problem of the lift force on a particular type of aerofoil was published some years earlier (in 1902). Furthermore, Chaplygin and Joukowskii were the first to describe (in 1910) the mapping of the exterior of a circle to the exterior region of a closed aerofoil-like profile; this conformal mapping is now commonly known as the Joukowskii transformation. In 1910 Chaplygin also published a paper in which he derived the equations for the forces and the moment on an aerofoil in a uniform (potential) flow in terms of contour integrals involving the complex flow potential; these results appeared at the same time as Blasius' similar expressions, which are now usually referred to in the literature. The analysis was later extended to various types of aerofoils, and in 1926 Chaplygin published a paper in which the generalization to unsteady flows was described. Also in 1910, Chaplygin published a paper in which he described the representation of a wing of finite span by a system of vortices – a similar investigation was published in the same year by the German scientist S. Finsterwalder. One year later (1911), Chaplygin presented the corresponding expressions for the lift and the induced drag forces. Because his theoretical results were contradicted by experimental results obtained in a wind tunnel that later appeared to have been too small, Chaplygin initially refrained from publishing his results in the scientific literature; this work was only published twenty years later. At approximately the same time, Ludwig Prandtl developed the closely related lifting-line theory, which became widely used in aerodynamics. For a historical account of the development of the theory of aerofoils, and in particular Chaplygin's contributions to this topic, the interested reader is referred to Satkevich (1923), Giacomelli & Pistolesi (1934) and Goldstein (1969). Chaplygin's pioneering work also included a study (manuscript dated 1921) of the interaction of two-dimensional vortex flow with some cylindrical obstacles (for which he derived analytical solutions) and of the stability of certain point vortex configurations.

It should be noted that Chaplygin's papers were all published in Russian, which is probably the reason why quite a few of his important contributions have escaped the attention of the wider fluid dynamics community. Although Chaplygin's papers were published quite some time ago, in our opinion they are still interesting and valuable for the present-day fluid dynamicist.

People interested in visiting Ranenburg, Chaplygin's place of birth, will not be able to find this town on the map: today this town bears the name Chaplygin, in honour of the great scientist that was born there in 1869.

REFERENCES

AREF, H. ROTT, N. & THOMANN, H. 1992 Gröbli's solution of the three-vortex problem: A case study of graduate work in the 1870's. *Ann. Rev. Fluid Mech.* **24**, 1–25.

ARNOL'D, V. I. 1965 Condition for nonlinear stability of stationary plane curvilinear flows of ideal fluid. *Sov. Math. Dokl.* **6**, 773–777.

AUERBACH, F. 1908 Wirbelbewegung. In *Handbuch der Physik* (2nd edn) (ed. A. Winkelmann), vol. 1, pp. 1047–1074. Barth.

BASSET, A. B. 1888 *A Treatise on Hydrodynamics*. Deighton Bell.

BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

CHAPLYGIN, S. A. 1899 On a pulsating cylindrical vortex. *Trans. Phys. Sect. Imperial Moscow Soc. Friends of Natural Sciences* **10**, N 1, 13–22. Also In *Collected Works*, 1948, vol. 2, pp. 138–154 (in Russian). German abstract: Über einen pulsierenden cylindrischen Wirbel. *Jahrbuch über die Fortschritte der Mathematik* (Jahrgang 1899), 1901, **30**, 683–684.

CHAPLYGIN, S. A. 1903 One case of vortex motion in fluid. *Trans. Phys. Sect. Imperial Moscow Soc. Friends of Natural Sciences* **11**, N 2, 11–14. Also In *Collected Works*, 1948, vol. 2, pp. 155–165 (in Russian).

COUDER, Y. & BASDEVANT, C. 1986 Experimental and numerical study of vortex couples in two-dimensional flows. *J. Fluid Mech.* **173**, 225–251.

DHANAK, M. R. & MARSHALL, M. P. 1993 Motion of an elliptic vortex under applied periodic strain. *Phys. Fluids* A **5**, 1224–1230.

DRAZIN, P. G. & HOWARD, L. N. 1966 Hydrodynamic stability of parallel flow of inviscid fluid. *Adv. Appl. Mech.* **9**, 1–89.

DRITSCHEL, D. G. 1990 The stability of elliptical vortices in an external straining flow. *J. Fluid Mech.* **210**, 223–261.

FLIERL, G. R. 1987 Isolated eddy models in geophysics. *Ann. Rev. Fluid Mech.* **19**, 493–530.

FLIERL, G. R., STERN, M. E. & WHITEHEAD, J. A. 1983 The physical significance of modons: laboratory experiments and general integral constraints. *Dyn. Atmos. Oceans* **7**, 233–264.

GIACOMELLI, R. & PISTOLESI, E. 1934 Historical sketch. In *Aerodynamic Theory. A General Review of Progress* (ed. W. F. Durand), 1963, vol. 1, pp. 305–394. Dover.

GOLDSTEIN S. 1969 Fluid mechanics in the first half of this century. *Ann. Rev. Fluid Mech.* **1**, 1–28.

GRIFFITHS, R. W. & LINDEN P. F. 1981 The stability of vortices in a rotating, stratified fluid. *J. Fluid Mech.* **105**, 283–316.

GRIGORYAN, A. T. 1965 Die Entwicklung der Hydrodynamik und Aerodynamik in den Arbeiten von N.J. Shukowski und S.A. Tschaplygin. *Schriftenreihe für Geschichte der Naturwissensch., Technik und Medizin* **2**, 39–62.

HEIJST, G. J. F. VAN & FLÓR, J. B. 1989*a* Dipole formation and collisions in a stratified fluid. *Nature* **340**, 212–215.

HEIJST, G. J. F. VAN & FLÓR, J. B. 1989*b* Laboratory experiments on dipole structures in a stratified fluid. In *Mesoscale/Synoptic Coherent Structures in Geophysical Turbulence* (ed. J. C. J. Nihoul & B. M. Jamart), pp. 609–626. Elsevier.

HELMHOLTZ, H. 1858 Über Integrale der hydrodynamischen Gleichungen welche den Wirbelbewegungen entsprechen. *J. Reine Angew. Math.* **55**, 25–55.

HILL, M. J. M. 1884 On the motion of fluid, part of which is moving rotationally and part irrotationally. *Phil. Trans. R. Soc. Lond.* A **175**, 363–410.

HOPFINGER, E. J. & HEIJST, G. J. F. VAN 1993 Vortices in rotating fluids. *Ann. Rev. Fluid Mech.* **25**, 241–289.

KIDA, S. 1981 Motion of an elliptic vortex in a uniform shear flow. *J. Phys. Soc. Japan* **50**, 3517–3520.

KIRCHHOFF, G. 1876 *Vorlesungen über mathematische Physik: Mechanik*. Teubner.

KLOOSTERZIEL, R. C. & HEIJST, G. J. F. VAN 1991 An experimental study of unstable barotropic vortices in a rotating fluid. *J. Fluid Mech.* **223**, 1–24.

LAMB, H. 1895 *Hydrodynamics* (2nd edn). Cambridge University Press.

LAMB, H. 1906 *Hydrodynamics* (3rd edn). Cambridge University Press.

LOITSYANSKII, L. G. 1966 *Mechanics of Liquids and Gases*. Pergamon.

LOVE, A. E. H. 1887 On recent English researches in vortex motion. *Math. Annalen* **30**, 326–344.

LOVE, A. E. H. 1893 On the stability of certain vortex motions. *Proc. London Math. Soc.* **25**, 18–34.

LOVE, A. E. H. 1901 Hydrodynamik: Theoretische Ausführungen. In *Encyklopädie der Mathematischen Wissenschaften* (ed. F. Klein & C. Müller), vol. 4, pp. 84–147. Teubner.

MAXWELL, J. C. 1855*a* Manuscript on the steady motion of an incompressible fluid. In *The Scientific Letters and Papers of James Clerk Maxwell* (ed. P. M. Harman), 1990, vol. 1, pp. 295–299. Cambridge University Press.

MAXWELL, J. C. 1855*b* Letter to William Thomson. In *The Scientific Letters and Papers of James Clerk Maxwell* (ed. P. M. Harman), 1990, vol. 1, pp. 309–313. Cambridge University Press.

MAXWELL, J. C. 1861*a* On physical lines of force. Part I. The theory of molecular vortices applied to magnetic phenomena. *Phil. Mag.* (4) **21**, 161–175.

MAXWELL, J. C. 1861*b* On physical lines of force. Part II. The theory of molecular vortices applied to electric currents. *Phil. Mag.* (4) **21**, 281–291, 338–348.

MOORE, D. W. & SAFFMAN, P. G. 1971 Structure of a line vortex in an imposed strain. In *Aircraft Wake Turbulence and its Detection* (ed. J. H. Olsen, A. Goldburg & M. Rogers), pp. 339–354. Plenum.

NGUYEN DUC, J.-M. & SOMMERIA, J. 1988 Experimental characterization of steady two-dimensional vortex couples. *J. Fluid Mech.* **192**, 175–192.

NYCANDER, J. & ISICHENKO, M. B. 1990 Motion of dipole vortices in a weakly inhomogeneous medium and related convective transport. *Phys. Fluids* B **2**, 2042–2047.

POLVANI, L. & WISDOM, J. 1990 Chaotic Lagrangian trajectories around an elliptical patch embedded in a constant and uniform background shear flow. *Phys. Fluids* A **2**, 123–126.

RANKINE, W. J. M. 1858 *A Manual of Applied Mechanics*. Griffin.

RICCA, R. L. 1991 Rediscovery of Da Rios equations. *Nature* **352**, 561–562.

SAFFMAN, P. G. 1992 *Vortex Dynamics*. Cambridge University Press.

SATKEVICH, A. A. 1923 *Aerodynamics as the Theoretical Basis of Aerial Flight*. Petrograd Institute of Ways of Communications Press.

SLEZKIN, N. G. 1988 Generalization of the Chaplygin problem of a cylindrical vortex. *Sov. Phys. Dokl.* **33**, 170–171.

STOKES, G. G. 1842 On the steady motion of incompressible fluids. *Trans. Camb. Phil. Soc.* **7**, 439–453.

VOROPAYEV, S. I. AFANASYEV, Y. D. & FILIPPOV, I. A. 1991 Horizontal jets and vortex dipoles in a stratified fluid. *J. Fluid Mech.* **227**, 543–566.

WIEN, W. 1900 *Lehrbuch der Hydrodynamik*. Hirzel.

YARMITSKII, A. G. 1992 A whirlwindlike Chaplygin vortex. *Fluid Dyn.* **27**, 489–494.